Statistical Orbital Error Calculation
Based on Sensitivity Analysis

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Outline

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- Uncertainty Propagation
- Orbit Equation

- Sensitivity Function
  - Orbital Error
  - In-track Error
  - Radial Error
  - Cross-track Error

- Unscented Transformation
- Numerical Example

- Conclusion
Introduction

Background
- 2009.02.10: Collision between Iridium 33 and Kosmos 2251
- Collision avoidance of satellites has become an important issue

Process

- Orbital error covariance
  - Correlate to the trajectory uncertainty
  - Probability density of trajectory is approximated by a Gaussian distribution with the mean and covariance
  - For the efficient avoidance maneuver, it is important to correctly predict the error covariance
Uncertainty Propagation

Uncertainty propagation

- Orbital error after specific time interval
- Orbital error is affected by initial position and velocity error

<table>
<thead>
<tr>
<th>Constant covariance</th>
<th>Linearized propagation model</th>
<th>Monte-Carlo simulation</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Assume that satellite has the same orbital error on whole orbit because initial velocity error is neglected</td>
<td>• Generally, dynamic model is linearized by Taylor series</td>
<td>• Provide true trajectory statistics</td>
</tr>
<tr>
<td></td>
<td>• Lead to significant errors when the system is highly nonlinear, when propagated over a long time interval, or when including a large initial error</td>
<td>• Require extensive computational resources</td>
</tr>
</tbody>
</table>

Sensitivity Analysis

- A kind of linearized propagation model

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Sensitivity function from orbit equation

Orbit equation of satellite

- Newtonian central gravity field \( \frac{\mu}{r^2} \)

\[
\ddot{r} - r \dot{\theta}^2 + \frac{\mu}{r^2} = 0
\]

- by Kepler’s 2\textsuperscript{nd} Law

\[
r^2 \dot{\theta} = r_0 U_0 \cos \gamma_0
\]

- Radius of target

\[
r(\phi) = \frac{\lambda r_0 \cos^2 \gamma_0}{1 - \cos \phi + \lambda \cos \gamma_0 (\phi + \gamma_0)} \quad (\lambda = \frac{r_0 U_0^2}{\mu})
\]

\[\lambda = 1: \text{circular orbit} \quad 1 < \lambda < 2: \text{elliptic orbit}\]

- flight time to target

\[
t_f(\phi) = \frac{1}{r_0 U_0 \cos \gamma_0} \int_0^\phi r^2(\theta) d\theta
\]

\[
r : \text{position of satellite} \quad \gamma_0 : \text{initial flight path angle}
\]

\[
\dot{\theta} : \text{angular velocity of satellite} \quad r_T : \text{target position}
\]

\[
\mu : \text{standard gravitational parameter} \quad \phi : \text{flight range angle}
\]
Orbital Error

Estimated states of satellite have none-zero error covariance

- Position error
  \[ \delta x_0, \delta y_0, \delta z_0 \sim N(0, \sigma^2) \]

- Velocity error
  \[ \delta U_0, \delta A_0, \delta \gamma_0 \sim N(0, \sigma^2) \]

Propagated orbital error due to initial states error

- After flight time (expected collision time) \( \bar{t}_f \)
- Define in the RIC (Radial, In-track, Cross-track) frame
  \[ \delta R, \delta I, \delta C \]

\( \bar{r}_0, \bar{r}_T, \bar{\phi} \) : nominal value
(without initial error)

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In-track Error (1)

- In-track error is proportional to $\delta \phi = \phi_f - \phi$ (assume circular orbit)

  - $\phi_f$: Flight range angle moved during flight time $t_f$ by initial states with error
  - $\delta \phi$ is divided to $\delta \phi_1$ and $\delta \phi_2$ ($\delta \phi = \delta \phi_1 + \delta \phi_2$)
    - $\delta \phi_1$: error due to $\delta x_0$
      $$\delta \phi_1 = \frac{1}{r_0} \delta x_0$$
    - $\delta \phi_2$: error due to $\delta z_0, \delta U_0, \delta \gamma_0$
      $$\delta t = \frac{\partial t}{\partial r_0} \delta r_0 + \frac{\partial t}{\partial U_0} \delta U_0 + \frac{\partial t}{\partial \gamma_0} \delta \gamma_0 + \frac{\partial t}{\partial \phi_0} \delta \phi_0$$
      $$\delta r_0 = -\delta z_0$$
    - Because time of two flights are the same: $\delta t = 0$
      $$\delta \phi_2 = -\frac{1}{\partial t / \partial \phi} \left( \frac{\partial t}{\partial r_0} \delta r_0 + \frac{\partial t}{\partial U_0} \delta U_0 + \frac{\partial t}{\partial \gamma_0} \delta \gamma' \right)$$
      $$(\delta \gamma' = \delta \gamma_0 + \delta \phi_1)$$

Calculus of variations

$$t_f(\phi) = \frac{1}{r_0 U_0 \cos \gamma_0} \int_0^\phi r^2(\theta)d\theta$$
In-track Error (2)

Cont’d

\[
\begin{align*}
\frac{\partial t}{\partial \phi} &= \frac{\bar{r}_f^2}{\bar{r}_0 \bar{U}_0 \cos \bar{y}_0} \\
\frac{\partial t}{\partial r_0} &= -\frac{\bar{t}_f}{\bar{r}_0} + \frac{2 \bar{\lambda}^2 \cos^3 \bar{y}_0}{\bar{U}_0} \int_0^{\bar{\varphi}} \frac{1 - \cos \theta + T}{T^3} d\theta \\
\frac{\partial t}{\partial U_0} &= -\frac{\bar{t}_f}{\bar{U}_0} + \frac{4 \bar{\lambda}^2 \bar{r}_0 \cos^3 \bar{y}_0}{\bar{U}_0^2} \int_0^{\bar{\varphi}} \frac{1 - \cos \theta}{T^3} d\theta \\
\frac{\partial t}{\partial \gamma_0} &= \bar{t}_f \tan \bar{y}_0 - \frac{2 \bar{\lambda}^2 \bar{r}_0 \cos^3 \bar{y}_0}{\bar{U}_0} \int_0^{\bar{\varphi}} \frac{2 \tan \bar{y}_0 (1 - \cos \theta) - \bar{\lambda} \sin \theta}{T^3} d\theta
\end{align*}
\]

T = 1 - \cos \theta + \bar{\lambda} \cos \bar{y}_0 \cos(\theta + \bar{y}_0)
Radial Error

Radius of position reached after flight time $\overline{t}_f$

$$r_T = \frac{\lambda r_0 \cos^2 \gamma_0}{1 - \cos \phi_T + \lambda \cos \gamma_0 (\phi_T + \gamma_0)}$$

$$\delta R = r_T - \overline{r}_T$$

by calculus of variations

$$\delta R = \frac{\partial r}{\partial r_0} \delta r_0 + \frac{\partial r}{\partial U_0} \delta U_0 + \frac{\partial r}{\partial \gamma_0} \delta \gamma_0 + \frac{\partial r}{\partial \phi} \delta \phi$$

$$\frac{\partial r}{\partial r_0} = \frac{\lambda \cos^2 \gamma_0 \{2 - 2 \cos \phi + \lambda \cos \gamma_0 \cos(\phi + \gamma_0)\}}{T^2}$$

$$\frac{\partial r}{\partial U_0} = \frac{2 \lambda r_0 \cos^2 \gamma_0 (1 - \cos \phi)}{U_0 T^2}$$

$$\frac{\partial r}{\partial \gamma_0} = -\frac{\lambda r_0 \cos \gamma_0 \{2 \sin \gamma_0 (1 - \cos \phi) - \lambda \cos \gamma_0 \sin \phi\}}{T^2}$$

$$\frac{\partial r}{\partial \phi} = -\frac{\lambda r_0 \cos^2 \gamma_0 \{\sin \phi - \lambda \cos \gamma_0 \sin(\phi + \gamma_0)\}}{T^2}$$

$$T = 1 - \cos \phi + \lambda \cos \gamma_0 \cos(\phi + \gamma_0)$$

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Cross-track Error

- **Cross-track error is proportional to** $\delta A$
  
  $$\delta C \approx \frac{\delta r}{r_0} \delta A \sin \phi = \frac{\delta r}{r_0} (\delta A_0 + \delta A_1) \sin \phi$$

- $\delta A_0$: Initial azimuth angle error
- $\delta A_1$: Due to lateral position error ($\delta y_0$)
  - $\delta y_0$: Normal to the nominal plane
  - Rotation of local NED frame (O frame -> O’ frame)
  - Arise normal component of velocity ($\delta U_y$) with respect to the nominal plane

\[
\beta \approx \frac{\delta y_0}{r_0 \sin \phi} \quad \text{assume that } \beta \text{ is small}
\]

\[
\delta U_y = U_0 \cos \phi \sin \beta \approx U_0 \beta \cos \phi
\]

\[
\delta A_1 \approx \frac{\delta U_y}{U_0} = \beta \cos \phi = \frac{\delta y_0}{r_0} \cot \phi
\]
Unscented Transformation

Simple sensitivity analysis of just one sigma value is not the same with Monte-Carlo simulation of random variables with one sigma square covariance.

- In the nonlinear system, the propagation of error distribution is nonlinear

Unscented Transformation (UT)

- Gaussian modeled random variable is nonlinearily transformed by UT
- Guarantee an accuracy to the second moment of Gaussian distribution
- Possible to compute much faster than MC simulation

the samples (sigma points) are not selected at random but rather according to a specific and deterministic scheme
Numerical Example (1)

**LEO satellite**
- Newtonian central gravity field
- Neglect the other effects except the gravity of the earth

<table>
<thead>
<tr>
<th>Latitude(deg)</th>
<th>Longitude(deg)</th>
<th>Altitude(km)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-57.9</td>
<td>164.2</td>
<td>700</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Speed(m/s)</th>
<th>FPA(deg)</th>
<th>Azimuth(deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>7504.3</td>
<td>0</td>
<td>-162.2</td>
</tr>
</tbody>
</table>

Position error covariance

\[
\sigma^2_{\delta x_0}(m^2) = 200^2 \\
\sigma^2_{\delta y_0}(m^2) = 10^2 \\
\sigma^2_{\delta z_0}(m^2) = 10^2 \\
\]

Velocity error covariance

\[
\sigma^2_{\delta U_0}(m^2/s^2) = 5^2 \\
\sigma^2_{\delta V_0}(deg^2) = 0.1^2 \\
\sigma^2_{\delta A_0}(deg^2) = 0.1^2 \\
\]

\[P_T = [-0.9 \text{deg}, -0.4 \text{deg}, 906 \text{km}]\]
### Numerical Example (2)

#### Single simulation vs. Sensitivity analysis
- with the same initial error ($1\sigma$)

<table>
<thead>
<tr>
<th></th>
<th>$\delta R(km)$</th>
<th>$\delta I(km)$</th>
<th>$\delta C(km)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Single simulation</td>
<td>25.060</td>
<td>-51.705</td>
<td>10.578</td>
</tr>
<tr>
<td>Sensitivity analysis</td>
<td>25.039</td>
<td>-51.881</td>
<td>10.531</td>
</tr>
<tr>
<td>Sim.-Sens.</td>
<td>0.021</td>
<td>0.176</td>
<td>0.047</td>
</tr>
<tr>
<td>(Sim.-Sens.)/Sim.</td>
<td>0.09%</td>
<td>0.34%</td>
<td>0.45%</td>
</tr>
</tbody>
</table>

#### MC simulation vs. Sensitivity analysis by using UT
- with the same initial error covariance
- No. of iteration : 300

<table>
<thead>
<tr>
<th></th>
<th>$\delta R(km)$</th>
<th>$\delta I(km)$</th>
<th>$\delta C(km)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>MC simulation</td>
<td>17.857</td>
<td>40.786</td>
<td>10.630</td>
</tr>
<tr>
<td>Sensitivity analysis by using UT</td>
<td>17.810</td>
<td>40.100</td>
<td>10.536</td>
</tr>
<tr>
<td>MC-Sens.</td>
<td>0.048</td>
<td>0.686</td>
<td>0.094</td>
</tr>
<tr>
<td>(MC-Sens.)/MC</td>
<td>0.27%</td>
<td>1.68%</td>
<td>0.88%</td>
</tr>
</tbody>
</table>
Conclusions

In order to calculate the orbital error statistics, we suggest the linearized error propagation model called sensitivity function.

Suggested model has less affect than the previous linearized propagation model related to propagated time interval.

Because orbital error propagation is the nonlinear transformation, we use the unscented transformation to represent more quickly and accurately the error distribution statistics.

Sensitivity analysis result is similar to MC simulation result, but it does not have enough accuracy for real-time collision prediction of satellite.
Thank you!!