Statistical Orbital Error Calculation Based on Sensitivity Analysis

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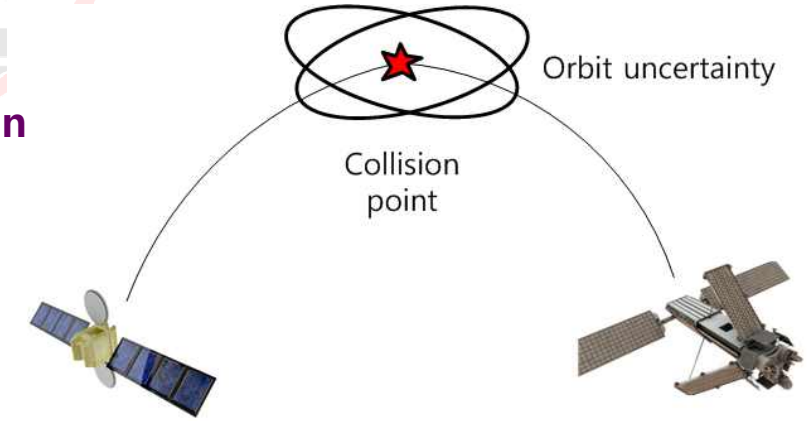
Introduction

Background

- 2009.02.10. : Collision between Iridium 33 and Kosmos 2251
- Collision avoidance of satellites has become an important issue
- Process

orbital error covariance → collision probability → collision avoidance

- Orbital error covariance
  - Correlate to the trajectory uncertainty
  - Probability density of trajectory is approximated by a Gaussian distribution with the mean and covariance
  - For the efficient avoidance maneuver, it is important to correctly predict the error covariance



Uncertainty Propagation

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Uncertainty propagation

- Orbital error after specific time interval
- Orbital error is affected by initial position and velocity error

Constant covariance	Linearized propagation model	Monte-Carlo simulation
• Assume that satellite has the same orbital error on whole orbit because initial velocity error is neglected	• Generally, dynamic model is linearized by Taylor series • Lead to significant errors when the system is highly nonlinear, when propagated over a long time interval, or when including a large initial error	• Provide true trajectory statistics • Require extensive computational resources

Sensitivity Analysis

- A kind of linearized propagation model

Orbit Equation

Sensitivity function from orbit equation

- **Orbit equation of satellite**
  - **Newtonian central gravity field($\frac{\mu}{r^2}$)**

$$\ddot{r} - r\dot{\theta}^2 + \frac{\mu}{r^2} = 0$$

  - **by Kepler's 2nd Law**

$$r^2\dot{\theta} = r_0 U_0 \cos\gamma_0$$

  - **Radius of target**

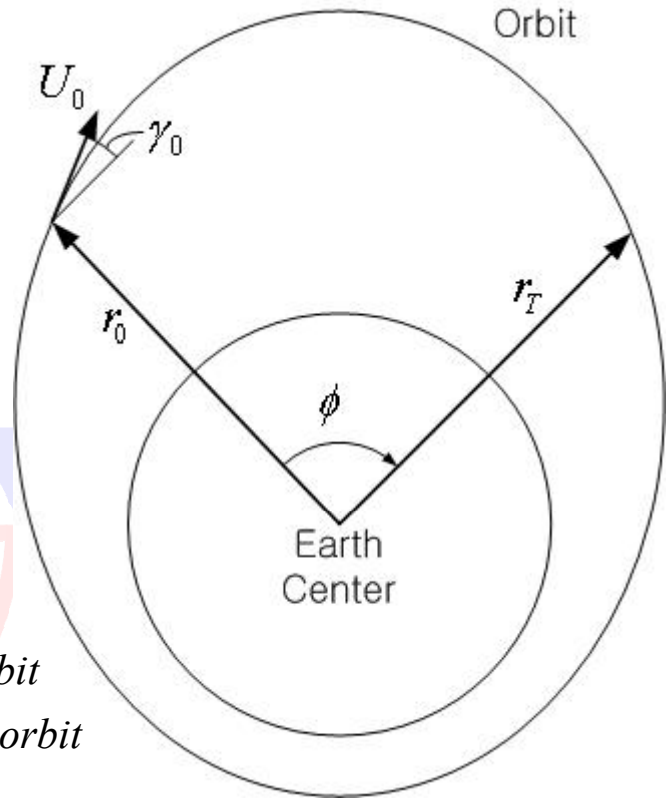
$$r(\phi) = \frac{\lambda r_0 \cos^2\gamma_0}{1 - \cos\phi + \lambda\cos\gamma_0(\phi + \gamma_0)} \qquad (\lambda = \frac{r_0 U_0^{\,2}}{\mu})$$

$$\lambda = 1 : circular\ orbit$$

$$1 < \lambda < 2 : elliptic\ orbit$$

  - **flight time to target**

$$t_f(\phi) = \frac{1}{r_0 U_0 \cos\gamma_0}\int_0^{\phi} r^2(\theta)d\theta$$



r_0 : *initial position*

U_0 : *initial velocity*

γ_0 : *initial flight path angle*

r_T : *target position*

ϕ : *flight range angle*

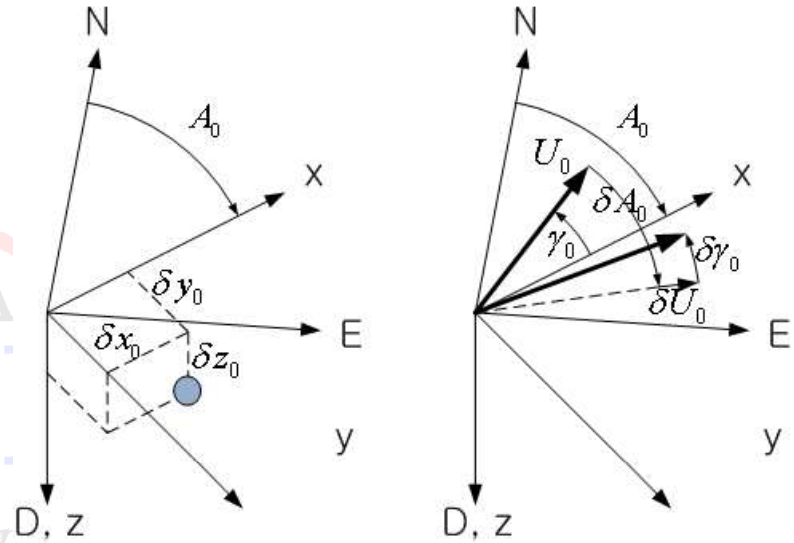
r : *position of satellite*

$\dot{\theta}$: *angular velocity of satellite*

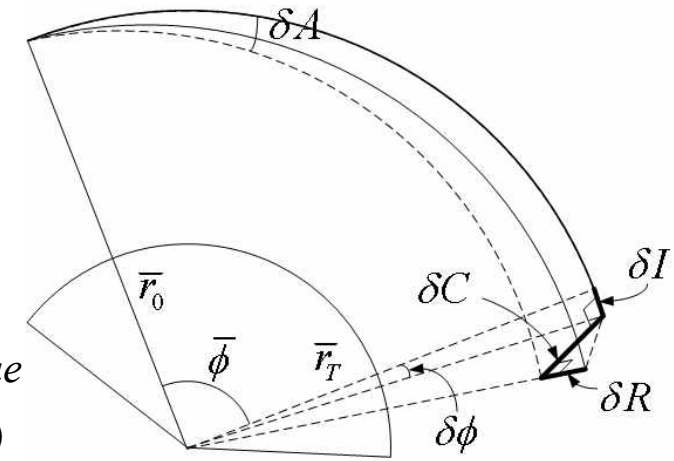
μ : *standard gravitational parameter*

Orbital Error

- **Estimated states of satellite have none-zero error covariance**
  - **Position error**
    $\delta x_0, \delta y_0, \delta z_0 \sim N(0, \sigma^2)$
  - **Velocity error**
    $\delta U_0, \delta A_0, \delta \gamma_0 \sim N(0, \sigma^2)$



- **Propagated orbital error due to initial states error**
  - **After flight time(expected collision time)** $\bar{t}_f$
  - **Define in the RIC(Radial, In-track, Cross-track) frame**
    $\delta R, \delta I, \delta C$

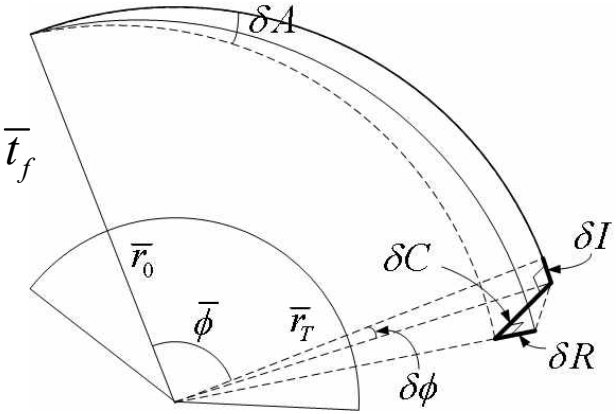


$\bar{r}_0, \bar{r}_T, \bar{\phi}$: *nominal value*
(*without initial error*)

In-track Error (1)

- **In-track error is proportional to $\delta\phi(=\phi_T-\bar{\phi})$**
  $\delta I \approx \bar{r}_T\delta\phi$ (assume circular orbit)
  - ϕ_T **: Flight range angle moved during flight time $\bar{t}_f$ by initial states with error**
  - $\delta\phi$ **is divided to $\delta\phi_1$ and $\delta\phi_2$** $(\delta\phi=\delta\phi_1+\delta\phi_2)$
    - $\delta\phi_1$ **: error due to δx_0**

$$\boxed{\delta\phi_1=\frac{1}{\bar{r}_0}\delta x_0}$$



    - $\delta\phi_2$ **: error due to $\delta z_0, \delta U_0, \delta\gamma_0$**

$$\delta t=\frac{\partial t}{\partial r_0}\delta r_0+\frac{\partial t}{\partial U_0}\delta U_0+\frac{\partial t}{\partial \gamma_0}\delta\gamma_0+\frac{\partial t}{\partial \phi_0}\delta\phi_0$$

$$(\delta r_0=-\delta z_0)$$

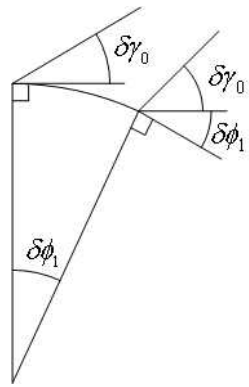
calculus of variations

$$t_f(\phi)=\frac{1}{r_0U_0\cos\gamma_0}\int_0^{\phi}r^2(\theta)d\theta$$

- **Because time of two flights are the same :** $\delta t=0$

$$\boxed{\delta\phi_2=-\frac{1}{\partial t/\partial\phi}\left(\frac{\partial t}{\partial r_0}\delta r_0+\frac{\partial t}{\partial U_0}\delta U_0+\frac{\partial t}{\partial \gamma_0}\delta\gamma'\right)}$$

$$(\delta\gamma'=\delta\gamma_0+\delta\phi_1)$$



In-track Error (2)

- **Cont'd**

$$\frac{\partial t}{\partial \phi} = \frac{\bar{r}_T^{\,2}}{\bar{r}_0 \bar{U}_0 \cos\bar{\gamma}_0}$$

$$\frac{\partial t}{\partial r_0} = -\frac{\bar{t}_f}{\bar{r}_0} + \frac{2\bar{\lambda}^2 \cos^3\bar{\gamma}_0}{\bar{U}_0}\int_0^{\bar{\phi}} \frac{1-\cos\theta + T}{T^3} d\theta$$

$$\frac{\partial t}{\partial U_0} = -\frac{\bar{t}_f}{\bar{U}_0} + \frac{4\bar{\lambda}^2 \bar{r}_0 \cos^3\bar{\gamma}_0}{\bar{U}_0^{\,2}}\int_0^{\bar{\phi}} \frac{1-\cos\theta}{T^3} d\theta$$

$$\frac{\partial t}{\partial \gamma_0} = \bar{t}_f \tan\bar{\gamma}_0 - \frac{2\bar{\lambda}^2 \bar{r}_0 \cos^3\bar{\gamma}_0}{\bar{U}_0}\int_0^{\bar{\phi}} \frac{2\tan\bar{\gamma}_0(1-\cos\theta) - \bar{\lambda}\sin\theta}{T^3} d\theta$$

$$T = 1 - \cos\theta + \bar{\lambda}\cos\bar{\gamma}_0 \cos(\theta + \bar{\gamma}_0)$$

Radial Error

- **Radius of position reached after flight time $\bar{t}_f$**

$$r_T = \frac{\lambda r_0 \cos^2 \gamma_0}{1 - \cos\phi_T + \lambda \cos\gamma_0 (\phi_T + \gamma_0)} \qquad \delta R = r_T - \bar{r}_T$$

- **by calculus of variations**

$$\boxed{\delta R = \frac{\partial r}{\partial r_0}\delta r_0 + \frac{\partial r}{\partial U_0}\delta U_0 + \frac{\partial r}{\partial \gamma_0}\delta\gamma' + \frac{\partial r}{\partial \phi}\delta\phi}$$

$$\frac{\partial r}{\partial r_0} = \frac{\bar{\lambda}\cos^2\bar{\gamma}_0\{2 - 2\cos\bar{\phi} + \bar{\lambda}\cos\bar{\gamma}_0\cos(\bar{\phi} + \bar{\gamma}_0)\}}{T^2}$$

$$\frac{\partial r}{\partial U_0} = \frac{2\bar{\lambda}\bar{r}_0\cos^2\bar{\gamma}_0(1 - \cos\bar{\phi})}{\bar{U}_0 T^2}$$

$$\frac{\partial r}{\partial \gamma_0} = -\frac{\bar{\lambda}\bar{r}_0\cos\bar{\gamma}_0\{2\sin\bar{\gamma}_0(1 - \cos\bar{\phi}) - \bar{\lambda}\cos\bar{\gamma}_0\sin\bar{\phi}\}}{T^2}$$

$$\frac{\partial r}{\partial \phi} = -\frac{\bar{\lambda}\bar{r}_0\cos^2\bar{\gamma}_0\{\sin\bar{\phi} - \bar{\lambda}\cos\bar{\gamma}_0\sin(\bar{\phi} + \bar{\gamma}_0)\}}{T^2}$$

$$T = 1 - \cos\bar{\phi} + \bar{\lambda}\cos\bar{\gamma}_0\cos(\bar{\phi} + \bar{\gamma}_0)$$

Cross-track Error

- **Cross-track error is proportional to δA**

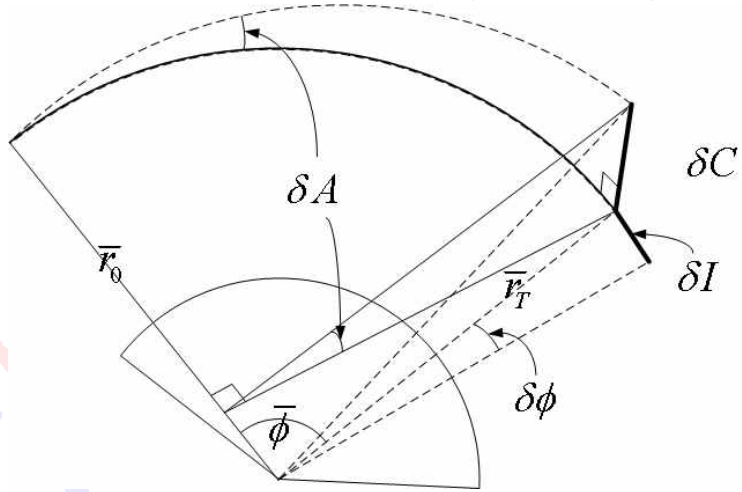
$$\delta C \approx \bar{r}_T \delta A \sin\bar{\phi} = \bar{r}_T(\delta A_0 + \delta A_1)\sin\bar{\phi}$$

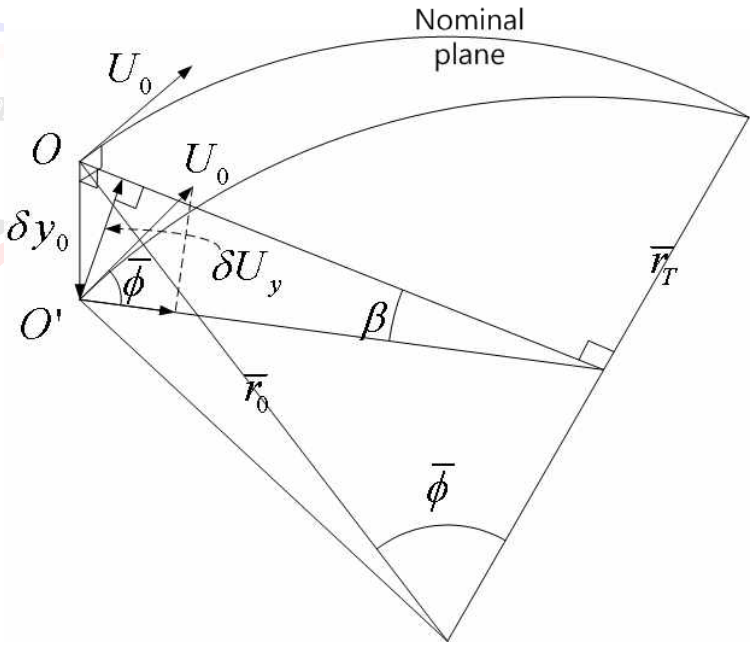
  - δA_0 **: Initial azimuth angle error**
  - δA_1 **: Due to lateral position error(δy_0)**
    - δy_0 **: Normal to the nominal plane**
    - **Rotation of local NED frame (O frame -> O' frame)**
    - **Arise normal component of velocity(δU_y) with respect to the nominal plane**

$$\beta \approx \frac{\delta y_0}{\bar{r}_0 \sin\bar{\phi}} \qquad \textit{assume that } \beta \textit{ is small}$$

$$\delta U_y = U_0 \cos\phi \sin\beta \approx U_0 \beta \cos\phi$$

$$\boxed{\delta A_1 \approx \frac{\delta U_y}{U_0} = \beta\cos\phi = \frac{\delta y_0}{r_0}\cot\phi}$$

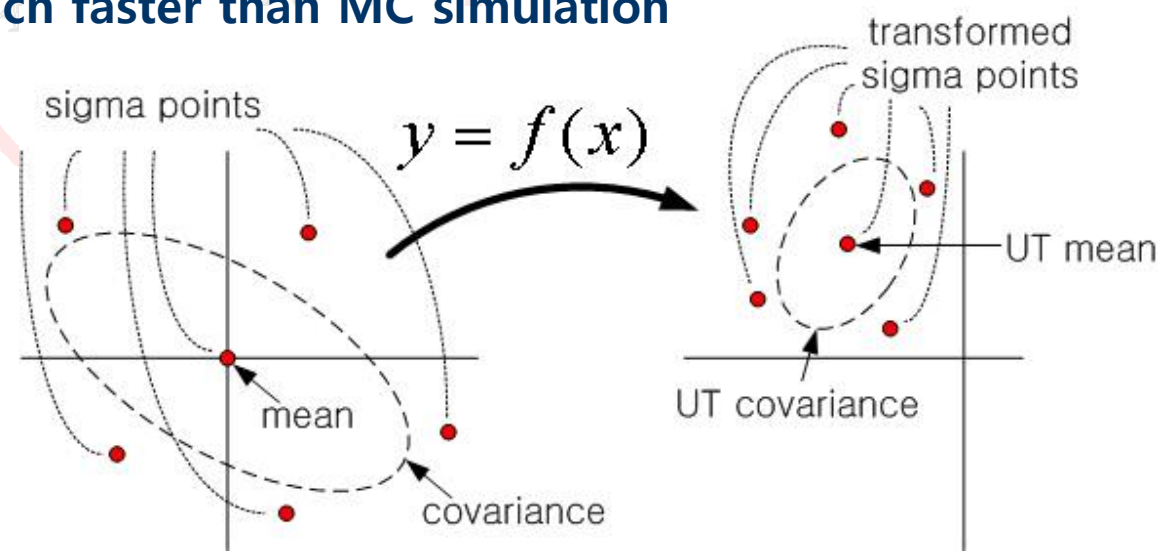




Unscented Transformation

- Simple sensitivity analysis of just one sigma value is not the same with Monte-Carlo simulation of random variables with one sigma square covariance.
  - In the nonlinear system, the propagation of error distribution is nonlinear
- Unscented Transformation(UT)
  - Gaussian modeled random variable is nonlinearly transformed by UT
  - Guarantee an accuracy to the second moment of Gaussian distribution
  - Possible to compute much faster than MC simulation

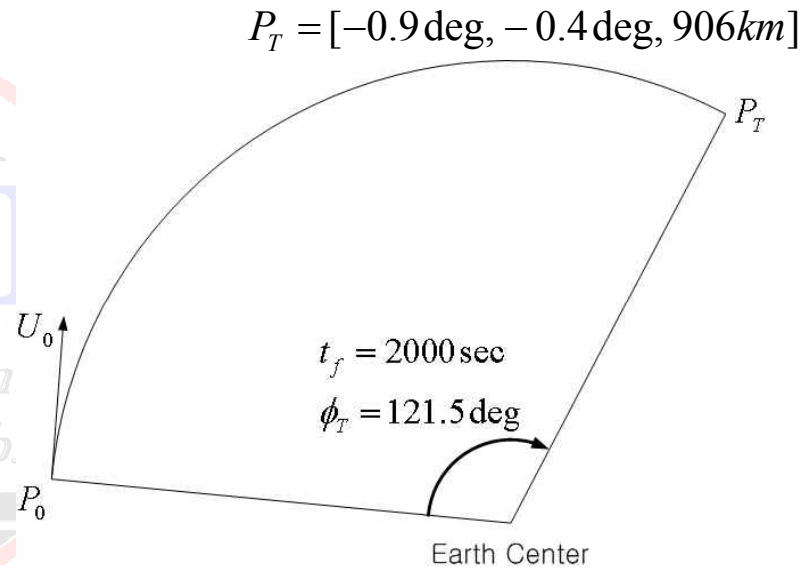
the samples(sigma points) are not selected at random but rather according to a specific and deterministic scheme



Numerical Example (1)

- **LEO satellite**
  - **Newtonian central gravity field**
  - **Neglect the other effects except the gravity of the earth**

Initial Position		
Latitude(deg)	Longitude(deg)	Altitude(km)
-57.9	164.2	700
Initial Velocity		
Speed(m/s)	FPA(deg)	Azimuth(deg)
7504.3	0	-162.2
Position error covariance		
$\sigma^2\_\delta x_0(m^2)$	$\sigma^2\_\delta y_0(m^2)$	$\sigma^2\_\delta z_0(m^2)$
200^2	10^2	10^2
Velocity error covariance		
$\sigma^2\_\delta U_0(m^2/s^2)$	$\sigma^2\_\delta \gamma_0(\deg^2)$	$\sigma^2\_\delta A_0(\deg^2)$
5^2	0.1^2	0.1^2



Numerical Example (2)

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- **Single simulation vs. Sensitivity analysis**
  - **with the same initial error (1σ)**

	$\delta R(km)$	$\delta I(km)$	$\delta C(km)$
Single simulation	25.060	-51.705	10.578
Sensitivity analysis	25.039	-51.881	10.531
Sim.-Sens.	0.021	0.176	0.047
(Sim.-Sens.)/Sim.	0.09%	0.34%	0.45%

- **MC simulation vs. Sensitivity analysis by using UT**
  - **with the same initial error covariance**
  - **No. of iteration : 300**

	$\delta R(km)$	$\delta I(km)$	$\delta C(km)$
MC simulation	17.857	40.786	10.630
Sensitivity analysis by using UT	17.810	40.100	10.536
MC-Sens.	0.048	0.686	0.094
(MC-Sens.)/MC	0.27%	1.68%	0.88%

Conclusions

- In order to calculate the orbital error statistics, we suggest the linearized error propagation model called sensitivity function

- Suggested model has less affect than the previous linearized propagation model related to propagated time interval

- Because orbital error propagation is the nonlinear transformation, we use the unscented transformation to represent more quickly and accurately the error distribution statistics

- Sensitivity analysis result is similar to MC simulation result, But it does not have enough accuracy for real-time collision prediction of satellite

Thank you!!